Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. State the definition of a convergent series. Find all positive values of $b$ for which the series $\sum_{n=1}^{\infty} b^{\log n}$ converges.

2. Let $f_n(x) = \sin nx$ for $n = 1, 2, \ldots$ and $x \in [0, 2\pi]$. Prove that $\{f_n\}$ does not contain a subsequence which converges pointwise on $[0, 2\pi]$.

3. State the definition of a compact space. Let $f : X \rightarrow Y$ be continuous, where $X$ and $Y$ are metric spaces. Prove that $f(X)$ is compact if $X$ is compact.

4. Suppose that $f$ is a uniformly continuous mapping of a metric space $X$ into a metric space $Y$. Prove that $\{f(x_n)\}$ is a Cauchy sequence in $Y$ for every Cauchy sequence $\{x_n\}$ in $X$.

5. Let $f_n(x) = nx^2/(1 + nx)$ for $x \in [0, 1]$.
   (A) Compute $\lim_{n \to \infty} f_n(x)$.
   (B) Is the convergence uniform? Prove your assertion.

6. Suppose $f'$ is continuous on $[a, b]$. Let $\varepsilon > 0$. Prove that there exists a $\delta > 0$ such that
   $$|(f(t) - f(x))/(t - x) - f'(x)| < \varepsilon$$
   whenever $0 < |t - x| < \delta$ and $a \leq x, t \leq b$.

7. Let $A$ be a dense subset of a metric space. Suppose $U$ is an open set. Prove that $U \subset (A \cap \overline{U})$, where $(A \cap \overline{U})$ is the closure of $A \cap U$.

8. State and prove Fatou’s lemma. Show that the inequality may be strict.

9. Let $f$ be Lebesgue integrable on $\mathbb{R}$ and suppose that $\int_I f \, dx = 0$ for every interval $I$. Prove that $f = 0$ a.e..