All work must be done in a neat and logical fashion. Give reasons for your steps, do not leave gaps in your arguments. Put your name on each sheet of paper and start each problem on a new sheet. There are 9 problems.

1. Let $f$ be continuous on the open interval $(a, b)$. Suppose $f(a+\epsilon)$ and $f(b-\epsilon)$ exist. Prove that $f$ is uniformly continuous on $(a, b)$.

2. Let $f$ be differentiable in $(a, \infty)$. Suppose $f(x) \rightarrow A$ as $x \rightarrow \infty$. Show that $\frac{f(x)}{x} \rightarrow A$ as $x \rightarrow \infty$.

3. Define $a_1 = 1$, and $a_{n+1} = \sqrt{2a_n}$. Prove that $\lim a_n$ exists and find this limit.

4. Let $g$ be a continuous function on $[0, 1]$, with $g(0) = 0$. Prove that $\left\{ x^n g(x) \right\}_{m=1}^{\infty}$ converges uniformly on $[0, 1]$.

5. Prove or disprove that every closed subset of the real line is a countable union of compact sets.

6. Let $f$ be a real valued measurable function. If $E$ is a Borel set, prove $f^{-1}(E)$ is measurable.

7. Let $E$ be a compact set and suppose $\{f_n\}$ is a sequence of continuous functions on $E$. If $f_{n+i} \leq f_m$ on $E$ and $f(x) = \lim f_m(x)$ is continuous, prove that the convergence is uniform on $E$. Show by an example that the uniform on $E$ is essential.
(8) Suppose \( \{f_n\} \) is a sequence of measurable functions on the measurable set \( E \).

Let \( A = \{ x \in E : \lim_{n \to \infty} f_n(x) \text{ exists} \} \). Prove that \( A \) is measurable.

(9) Let \( f \) be a non-negative integrable function.

Prove that for every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( \int_A f \, d\mu < \varepsilon \), whenever \( \mu(A) < \delta \).

[Hint: First prove for simple functions.]