Be sure to put each problem on a separate sheet. Do not leave any gaps in your proofs. State all theorems you use in the course of your proof.

1. Let \((x_n)\) be a sequence of real numbers such that \(\lim x_n = x\). Suppose that \(x_m > x\) for each \(m\). Prove that one can change the order of the terms in \((x_n)\) to obtain a decreasing sequence.
   
   [Hint: Let \(\alpha > 0\). How many \(x_m\) are outside the interval \((x, x+\alpha)\)?]

2. Let \(f : \mathbb{R} \to \mathbb{R}\) be a function and let \(x_0 \in \mathbb{R}\). Assume that for every decreasing sequence \((x_m)\) with \(x_m \not= x\) and \(x_m > x_0\), the sequence \((f(x_m))\) is Cauchy.
   
   Prove: (a) The limit \(l = \lim f(x_m)\) is independent of the sequence.
   
   (b) \(\lim_{x \to x_0} f(x) = l\).

   [Hint: Given two such sequences \((x_n)\) and \((y_n)\), form a certain subsequence \(z_m \not= x_0\) so that \(z_{2m}\) is a subsequence of \((x_n)\) and \(z_{2m+1}\) is a subsequence of \((y_n)\).]

3. Prove: (1) Every bounded sequence in \(\mathbb{R}\) has a convergent subsequence.
   
   (2) Let \((x_n)\) be a bounded sequence in \(\mathbb{R}\).
   
   Suppose all convergent subsequences have the same limit point. Prove \(\lim x_n\) exists.

4. Prove that the continuous image of a connected set is connected.
5) Suppose $a_m > 0$ and $b_m > 0$, $n = 1,2,...$. Suppose $\sum a_m < \infty$ and $\sum b_m = \infty$. Prove or disprove $\sum \frac{a_m}{b_m} < \infty$.

6) Suppose $f$ is differentiable on $(0,\infty)$ and $\lim_{x \to \infty} (f(x) + f'(x)) = A$. Prove $\lim_{x \to \infty} f(x) = A$ and $\lim_{x \to \infty} f'(x) = 0$. [Hint: Use L'Hôpital on $\frac{h(x)}{g(x)}$ where $h(x) = e^x f(x)$. You determine what $g$ should be and supply all details.]

7) Let $f: \mathbb{R} \to \mathbb{R}$ be a bounded Lebesgue measurable function. Prove or disprove that there exists a sequence of simple measurable functions which converge uniformly to $f$.

8) State and prove Fatou's Theorem and show that the inequality may be strict.

9) Let $(f_m)$ be a sequence of Lebesgue integrable functions defined on $\mathbb{R}$ such that $f_m(x) \to 0$ uniformly on $\mathbb{R}$. Prove or disprove that $\lim_{m \to \infty} \int f_m \, dm = 0$, where $\int$ denotes Lebesgue measure.