First Year Exam in Analysis
May, 1978

Put each problem on a separate page. Justify all your steps. Present all work in a neat and logical fashion.

1. Let \( f : (0,1] \to [0,1] \) be defined by
   \[ f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{m}, & \text{if } x = \frac{m}{m}, \text{ with } \gcd(m, m) = 1. \end{cases} \]
   Find all points \( x \in (0,1] \) such that \( f \) is continuous at \( x \).

2. Let \( A \subset \mathbb{R}^2 \). Call a point \( p \in \mathbb{R}^2 \) a condensation point of \( A \) if every neighborhood of \( p \) contains uncountably many points of \( A \). Let \( P = \{ q \in \mathbb{R}^2 : q \text{ is a condensation point of } A \} \).
   Prove that \( P \) is a closed set.

3. Let \( \{f_m\} \) be a sequence of continuous real valued functions defined on a compact metric space \( K \). Suppose \( \{f_m\} \) converges pointwise to a continuous function \( f \) on \( K \) and \( f_m(x) \leq f_{m+1}(x) \)
   for all \( x \in K \) and \( m = 1, 2, \ldots \).
   Prove that \( \{f_m\} \) converges uniformly to \( f \) on \( K \).

4. If \( c_0 + \frac{c_1}{2} + \ldots + \frac{c_{m-1}}{m} + \frac{c_m}{m+1} = 0 \), where \( c_0, \ldots, c_m \) are real constants, prove that the equation \( c_0 + c_1 x + \ldots + c_{m-1} x^{m-1} + c_m x^m = 0 \)
   has at least one real root between 0 and 1.

5. Suppose \( \sum a_m \) is a convergent series, where \( a_m \geq 0 \) for each \( m \). Discuss the convergence of \( \sum \frac{\sqrt{a_m}}{m} \).

6. Let \( f \) be Lebesgue integrable on \( \mathbb{R}^2 \) and suppose \( \int_A f \, dm = 0 \) for every measurable subset \( A \subset \mathbb{R}^2 \).
   What can you conclude about \( f \)?
Let $f$ be a real valued Lebesgue integrable function on $[0, \infty)$ such that $\int_0^t f(\xi) d\xi \geq 0$ for all $t \geq 0$. Is it true that $f \geq 0$ a.e.

Prove your answer.

Let $E \subset [0,1]$ be a measurable set and suppose $0 \leq \alpha \leq m(E)$. Prove that there exists a measurable set $F \subset E$ such that $m(F) = \alpha$.

(Hint: Examine the function on $[0,1]$, $f(x) = m((0,1] \cap E)$)

Let $(X,d)$ be a complete metric space, $f : X \to X$ and assume that there exists a $0 < k < 1$ such that $d(f(x), f(y)) \leq kd(x,y)$ for all $(x,y) \in X \times X$.

(a) Prove $d(x_{m+p}, x_m) \leq (k^{p-1} + \cdots + 1)k^m d(x_1, x_0) \leq \frac{k^m}{1-k} d(x_1, x_0)$ where $x_0 \in X$ and $x_{m+1} = f(x_m)$.

(b) Show that there exists exactly one $\bar{x} \in X$ such that $f(\bar{x}) = \bar{x}$. (Hint: Is $\{x_m\}$ a Cauchy sequence?)

(a) Show the existence of a bijective map $\psi$ from $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{Z}^+$ where $\mathbb{Z}$ is the set of integers by an explicit construction or picture of one.

(b) Consider the sequence $\{c_{m,m} : m, m \in \mathbb{Z}\}$ indexed by $(m,m) \in \mathbb{Z} \times \mathbb{Z}.$ Let $\psi$ be your map in (a). Suppose that $\sum c_{m,m}$ converges to some number in $\mathbb{R}$, but also assume that the series is NOT absolutely convergent.

\[
\left[ \sum_{(m,m)} c_{m,m} \right] \text{ means it is the series } \sum_{m=1}^{\infty} t_m, \text{ where } t_m = c_{\psi(m)}.
\]

Let $b \in \mathbb{R}$.

Show that there exists a bijective map $\psi_b : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^+$ such that $\sum_{\psi_b(m,m)} c_{m,m} = b$.

(c) Conclude directly from (b) that the set of bijective maps from $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^+$ is uncountable.