First year exam in Analysis
September 1997

Do each of the ten problems. Be sure to put each problem on a separate page. Print your name on each page handed in. Write complete sentences and use a complete and precise mathematical style. All work must be done in a neat and logical fashion in order to obtain full credit.

1. State and prove the monotone convergence theorem for nonnegative functions.

2. Let \( f(x) = \begin{cases} \ x^2 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases} \)

   a) Find the points of continuity of \( f \).
   b) Find the points where \( f \) is differentiable.

3. Prove that there is a unique number \( c > 1 \) such that \( \int_1^x \frac{dt}{\log t} = \log(\log x) + \sum_{n=1}^{\infty} \frac{(\log x)^n}{n(n!)} \), using the following steps:
   a) Prove that \( \sum_{n=1}^{\infty} \frac{(\log x)^n}{n(n!)} \) converges for \( x \geq 1 \);
   b) Prove that \( f(x) = \log(\log x) + \sum_{n=1}^{\infty} \frac{(\log x)^n}{n(n!)} \) is differentiable for \( x > 1 \) and that \( f'(x) = \frac{1}{\log x} \).
   c) Prove that \( f \) has a unique zero, \( c > 1 \).

4. Prove that a measurable function \( f \geq 0 \) is the limit of an increasing sequence of simple functions, and that a real, bounded, measurable function is the uniform limit of a sequence of simple functions.

5. Let \( f_n \) be a sequence of real function defined on the interval \([0,3]\) by

\[
f_n(x) = \begin{cases} 
  x & \text{if } 0 \leq x \leq \frac{1}{n} \\
-x + \frac{2}{n} & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\
 x - \frac{2}{n} & \text{if } \frac{2}{n} \leq x \leq 3 
\end{cases}
\]

and let \( f : [0,3] \rightarrow \mathbb{R} \) be defined by \( f(x) = x \). Does \( (f_n) \) converge pointwise to \( f \) on \([0,3]\)? Does \( (f_n) \) converge uniformly to \( f \) on \([0,3]\)?

6. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable function such that \( f'(t) \neq 1 \) for every \( t \in \mathbb{R} \). Prove that there is at most one point \( x \in \mathbb{R} \) with \( f(x) = x \). Give a complete statement of the theorems used in your proof.
7. Let \( f : [0, 1] \to \mathbb{R} \) be a monotone function. Show that for every \( n \) we have

\[
\left| \int_0^1 f(x) \, dx - \frac{1}{n} \sum_{k=1}^{n} f\left( \frac{k}{n} \right) \right| \leq \frac{1}{n} |f(1) - f(0)|.
\]

State the theorem of approximation of the integral by Riemann sums.

8. Let \( f : (a, b) \to \mathbb{R} \) be a monotone function. Prove that the set of points of discontinuity of \( f \) is at most countable.

9. Let \( E \subseteq \mathbb{R} \) be any set and \( f : E \to \mathbb{R} \) a monotonic function. Prove that if the range \( f(E) \) is an interval, then \( f \) is continuous on \( E \). Hint: prove separately left continuity and right continuity.

10. Give the definition of a compact set in a metric space.
    Let \( X \) be a compact metric space and \( f : X \to \mathbb{R} \) a continuous function.
    Show that \( f \) attains its supremum and its infimum.