Do each of the ten problems. Be sure to put each problem on a separate page. Print your name on each page handed in. All work must be done in a neat and logical fashion in order to obtain credit.

1. Let $K$ be a compact subset of a metric space $X$. Prove that $K$ is closed.

2. Let $Y$ be the metric space consisting of all continuous, real valued functions defined on $[0,1]$ with metric

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$ 

Let $p$ be a fixed real number. Let $E$ be the subset of $Y$ which consists of all $f \in Y$ such that $p$ is not in the range of $f$. Prove that $E$ is an open subset of $Y$.

3. Let $E \subset R$ be a segment (open interval), and suppose $f : E \to R$ is monotonically increasing. Suppose also that $f(E)$ (the range of $f$) is a segment. Prove that $f$ is continuous on $E$.

4. Suppose that $\{a_k\}$ is a sequence of non-zero real numbers, and

$$q = \lim_{k \to \infty} \frac{\log(\frac{1}{|a_k|})}{\log k}$$ exists

Prove that $\sum_{k=1}^{\infty} a_k$ converges absolutely if $q > 1$.

Hint: There is a real number $p$ with $1 < p < q$.

5. Let $g : R \to R$ be defined by

$$g(x) = \begin{cases} 
  x & \text{if } x < 0, \\
  x + 1 & \text{if } x \geq 0.
\end{cases}$$

Does there exist a differentiable function $f : R \to R$ such that $f'(x) = g(x)$ for each $x \in R$?

Give complete verification of your answer.

6. Let $f : [0,1] \to [0,\infty]$ be continuous. If $\int_0^1 f(x)dx = 0$, what can you say about $f$?

Give complete verification of your answer.
7. If

\[ I(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
1 & \text{if } x > 0, 
\end{cases} \]

if \{x_n\} is a sequence of distinct points of (a, b) and if \( \sum_{n=1}^{\infty} |c_n| \) converges, what can you say about the convergence of the series

\[ f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n), \quad a \leq x \leq b? \]

If \( x \in [a, b] \) and \( x \neq x_n \) for each \( n \), is \( f \) continuous at \( x \)? Give complete verification of your answers.

8. Let \( X \) be a measurable space, and let \( f \) be a real valued function defined on \( X \).

8a. Complete the following definition. The function \( f \) is said to be measurable if…….

8b. Prove that \( f \) is measurable if and only if for every open subset \( V \) of \( R \), \( f^{-1}(V) \) is measurable.

Hint: Any open subset of \( R \) is a countable union of segments (open intervals).

9. Let \( (X, m, \mu) \) be a measurable space, and let \( f \geq 0 \) be an integrable function (with respect to \( \mu \)). Suppose \( \int_X f \ d\mu = 0 \).

Prove that \( f = 0 \) almost everywhere (with respect to \( \mu \)).

10a. State and prove Fatou’s theorem.

10b. Show that the inequality appearing in the theorem may be strict.