DO EACH OF THE TEN PROBLEMS. Be sure to put each problem on a separate page. Print your name on every page handed in. All work must be done in a neat and logical fashion in order to obtain credit.

1. If \( \{f_n\} \) and \( \{g_n\} \) are sequences of real-valued functions on a set \( E \) which converge uniformly to bounded functions \( f \) and \( g \), respectively, show that \( \{f_ng_n\} \) converges uniformly on \( E \).

2. Let \( A \subseteq \mathbb{R} \) be a non-empty set which has the property that every sequence in \( A \) has a subsequence that converges to a point of \( A \). Show that \( A \) is bounded above and that the least upper bound of \( A \) is an element of \( A \).

3. Let \( m \) denote Lebesgue measure on the real line and let \( f: \mathbb{R} \to \mathbb{R} \) be a Lebesgue integrable function. Suppose that \( \int_E f \, dm = 0 \) for every measurable \( E \subseteq \mathbb{R} \). Show that \( f = 0 \) almost everywhere.

4(a) Suppose that \( \sum a_n \) is a series of non-negative real numbers whose partial sums are bounded. Prove that the series

\[
\sum_{n=1}^{\infty} \frac{a_n}{n}
\]

is convergent.

(b) Does the conclusion of (a) remain true if the non-negativity hypothesis on the \( a_n \)'s is removed? Justify your answer.

5. Suppose that \( p > 0 \). Show that

\[
\lim_{n \to \infty} \sqrt[n]{p} = 1.
\]

6. Suppose that \( \{f_n\} \) is a sequence of continuous functions \( f_n: [0,1] \to \mathbb{R} \) which converges uniformly to \( f: [0,1] \to \mathbb{R} \). Prove, starting from the definition of uniform convergence, that

\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx.
\]

7. Let \( \varphi \) be a finitely additive set function defined on a \( \sigma \)-ring \( \mathcal{R} \). Suppose that for any sequence of sets \( A_n \) from \( \mathcal{R} \) such that \( A_n \supseteq A_{n+1} \) for \( n = 1, 2, 3, \ldots \) and \( \cap_{n=1}^{\infty} A_n = \varnothing \), we have \( \lim_{n \to \infty} \varphi(A_n) = 0 \). Prove that \( \varphi \) is countably additive on \( \mathcal{R} \).
8(a) State the Generalized Mean Value Theorem.
(b) Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function which is differentiable at every point of $(0, 1)$. Show that for each $n = 1, 2, 3, \ldots$ there exists $c \in (0, 1)$ such that

$$f(1) - f(0) = \frac{f'(c)}{nc^{n-1}}.$$

9. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$h(x) = \begin{cases} 
0, & x < -1 \\
1, & -1 \leq x \leq 1 \\
0, & 1 < x
\end{cases}$$

Is there a differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f'(x) = h(x)$?

10. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 
\sum_{n=1}^{\infty} (1 + n^3x)^{-1}, & \text{if } x > 0, \\
0, & \text{if } x = 0.
\end{cases}$$

(a) Is $f$ continuous at $x = 0$?
(b) Determine the largest interval on which $f$ is continuous.
[Explain your answers to both questions carefully.]