DO EACH OF THE TEN PROBLEMS. Be sure to put each problem on a separate page. Print your name on every page handed in. All work must be done in a neat and logical fashion in order to obtain credit.

1. Let \( \{f_n\} \) be a sequence of Lebesgue integrable functions defined on \( \mathbb{R} \) such that

\[
\int_{\mathbb{R}} |f_n| \, dm \leq \frac{1}{n^2} \quad (n = 1, 2, \ldots)
\]

Prove that \( \sum_{n=1}^{\infty} f_n(x) \) converges for almost every \( x \in \mathbb{R} \).

2. Define

\[
E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!},
\]

for \( z \) complex. Prove that \( E(z + w) = E(z)E(w) \) for all \( z, w \in \mathbb{C} \). Justify all the steps in your argument.

3. A subset \( A \) of \( \mathbb{R}^n \) is convex if whenever \( x, y \in A \) and \( 0 < \lambda < 1 \) then \( \lambda x + (1 - \lambda)y \in A \). Prove that every convex subset of \( \mathbb{R}^n \) is connected.

4. Let \( A \) and \( B \) be Lebesgue measurable subsets of \([0, 1]\). Suppose that the Lebesgue measures of \( A \) and \( B \) are

\[
m(A) = \frac{1}{2},
\]

\[
m(B) = \frac{3}{4}.
\]

What are the maximum and minimum possible values of \( m(A \cap B) \)? Are all values between these extrema attained by \( m(A \cap B) \)?

5. Suppose that \( a_n \geq 0 \) for each \( n \) and that \( a_n \) is a decreasing sequence. Prove that \( \sum_{n=1}^{\infty} a_n \) converges if and only if

\[
\sum_{n=1}^{\infty} 3^n a_{3^n}
\]

converges.
6. Suppose that \( \phi: [a, b] \to \mathbb{R} \) is Riemann integrable and positive, and that \( f: [a, b] \to \mathbb{R} \) is continuous. Prove that there is a point \( x \in [a, b] \) such that
\[
\frac{\int_a^b f(t) \phi(t) \, dt}{\int_a^b \phi(t) \, dt} = f(x).
\]

7. Suppose that \( a < c < b \) and that \( f: (a, b) \to \mathbb{R} \) is a twice continuously differentiable function.
   (a) Compute
   \[
   \lim_{t \to 0} \frac{f(c + t) - f(c - t)}{2t}.
   \]
   (b) Compute
   \[
   \lim_{t \to 0} \frac{f(c + 2t) - 2f(c + t) + f(c)}{t^2}.
   \]

8. Suppose that \( a < b \) and that \( f: [a, b] \to \mathbb{R} \) is continuous. Prove that the image of \([a, b]\) under \( f \) is a closed and bounded interval.

9. Suppose that \( \{f_n(x)\} \) is a sequence of continuous functions and that \( \{f_n(x)\} \) converges uniformly to \( f(x) \) on \([0, 1]\). Prove that \( f(x) \) is continuous on \([0, 1]\).

10. Let
    \[
    f_n(x) = \frac{1}{nx + 1},
    \]
    for \( 0 < x < 1 \) and \( n = 1, 2, 3, \ldots \)
    (a) Show that \( f_n(x) \to 0 \) monotonically in \((0, 1)\).
    (b) Is the convergence uniform?