DO AT MOST EIGHT OF THE TEN PROBLEMS. Be sure to put each problem on a separate page. Print your name on each page handed in. All work must be done in a neat and logical fashion in order to obtain credit.

1. Fill in the blank with one of the three words “compact,” “bounded,” “closed.” If \( \{A_\alpha\} \) is a collection of subsets of the reals such that the intersection of every finite subcollection of \( \{A_\alpha\} \) is nonempty, then \( \cap_\alpha A_\alpha \) is nonempty.

   Give examples showing that neither of the choices you omitted would make the statement correct.

2. Let \( D \) be a bounded subset of the reals. If \( f : D \to \mathbb{R} \) is continuous, must \( f \) be uniformly continuous?

3. Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of real numbers with \( a_n > 0 \) and \( b_n \geq 0 \). If both the sequence \( \{\frac{b_n}{a_n}\} \) and the series \( \sum a_n \) converge, does the series \( \sum b_n \) converge?

4. Define \( f : \mathbb{R} \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
  x^2 \sin(1/x) + \frac{x}{2} & \text{if } x \neq 0 \\
  0 & \text{if } x = 0.
\end{cases}
\]

   (1) Find \( f'(0) \);
   (2) find \( f'(\frac{1}{2n\pi}) \);
   (3) Is there an interval containing 0 on which \( f \) is increasing?

5. Let \( \{f_n\} \) be a sequence of Riemann integrable functions on the interval \([0,1]\). Let

\[
F_n(x) = \int_0^x f_n(t)dt + f_n(0).
\]

Prove

(1) If \( \{f_n\} \) is uniformly bounded (i.e. \( |f_n(x)| \leq M \) for all \( x \in [0,1] \) and all \( n \in \mathbb{N} \)), then there exists a subsequence \( \{F_{n_k}\} \) which converges uniformly on \([0,1]\).

(2) If each \( f_n \) is bounded, but not necessarily uniformly bounded, then there exists a subsequence \( \{F_{n_k}\} \) which converges at each rational number \( q \) in \([0,1]\).

6. Let \( f_n(x) = \sin(nx), 0 \leq x \leq 2\pi, n = 1, 2, 3, \ldots \)

   (1) Find \( \int_0^{2\pi} |f_n - f_m|^2 dx \).
   (2) Does there exist a subsequence \( \{n_k\} \) such that \( \{\sin(n_kx)\} \) converges for every \( x \in [0,2\pi] \).
7. Find a closed subset of the reals with positive Lebesgue measure that does not intersect the rationals.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Let $m$ denote Lebesgue measure. Prove, for every $\epsilon > 0$ there exists a Lebesgue measurable set $A_\epsilon$, such that $m(A_\epsilon) < \infty$ and $\int_B |f| dm < \epsilon$, where $B = \mathbb{R} \setminus A_\epsilon$.

9. Let $\{f_n\}$ be a sequence of measurable functions. Prove the set of points $x$ at which $\{f_n(x)\}$ converges is a measurable set.

10. Show, if $f : [0, 1] \to \mathbb{R}$ is Riemann integrable and $f(g) = 0$ for every rational number $g \in [0, 1]$, then $\int_0^1 f dx = 0$. Is the hypothesis $f$ is Riemann integrable needed? What happens if Riemann integrable is replaced by Lebesgue integrable?