Be sure to put each problem on a separate page. Print your name on each page handed in. All work must be done in a neat and logical fashion in order to obtain credit. Each of the 10 problems is worth 10 points.

1. Let $X$ be a metric space.
   (a) Give the definition for a subset $E$ of $X$ to be compact.
   (b) Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of points in $X$. Suppose $\{x_n\}_{n=1}^{\infty}$ converges to $x_0$. Let $E = \{x_n : n = 0, 1, 2, \ldots\}$. Use the definition in part (a) to prove that $E$ is compact.

2. Let $X$ be a metric space.
   (a) Give the definition for a sequence $\{x_n\}_{n=1}^{\infty}$ from $X$ to be a Cauchy sequence.
   (b) Let $A$ be a subset of $X$ such that given any $\epsilon > 0$, $A$ can be covered by finitely many open balls in $X$ of radius $\epsilon$. Prove that every sequence of elements in $A$ has a Cauchy subsequence.

3. Let $D$ be a bounded subset of the real numbers. We do not assume that $D$ is open or closed. Let $f$ be a bounded continuous function defined on $D$. Is $f$ uniformly continuous on $D$?

4. Given a series $\sum a_n$, define
   \[ p_n = \frac{|a_n| + a_n}{2} \quad \text{and} \quad q_n = \frac{|a_n| - a_n}{2}. \]
   Show the series $\sum a_n$ converges absolutely if and only if both series $\sum p_n$ and $\sum q_n$ converge.

5. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers with $a_n > 0$ and $b_n \geq 0$ for all $n$. If both the sequence $\{\frac{b_n}{a_n}\}$ and the series $\sum a_n$ converge, does the series $\sum b_n$ converge?
6. Let $f_n(x)$ be a sequence of functions defined on a countable set $E = \{x_1, x_2, \ldots, x_n, \ldots\}$, which satisfies

$$|f_n(x_i)| \leq i, \text{ for all } n = 1, 2, \ldots \text{ and } i = 1, 2, \ldots .$$

Find a subsequence of $\{f_n\}$ which converges for every $x$ in $E$.

7. Let $\alpha$ be a monotonically increasing function on $[a, b]$. Suppose $x_0 \in (a, b)$ and $\alpha$ is continuous at $x_0$. Define $f$ on $[a, b]$ by $f(x_0) = 1$ and $f(x) = 0$ if $x \neq x_0$. Is $f$ integrable in the Riemann–Stieljes sense with respect to $\alpha$? If so, find the value of this integral, $\int_a^b f \, d\alpha$.

8. Let $\Sigma$ be a $\sigma$–ring of subsets of a set $X$. Let $\mu : \Sigma \to [0, \infty]$ be finitely additive. Suppose that if $A_n \in \Sigma$, $A_{n+1} \subseteq A_n$ and $\bigcap_{i=1}^{\infty} A_n$ is empty, then the sequence $\{\mu(A_n)\}_{i=1}^{\infty}$ converges to 0. Prove that $\mu$ is countably additive.

9. Suppose

1. $|f(x, y)| \leq 1$ if $0 \leq x \leq 1, 0 \leq y \leq 1$,
2. for fixed $x$, $f(x, y)$ is a continuous function of $y$,
3. for fixed $y$, $f(x, y)$ is a continuous function of $x$.

Put

$$g(x) = \int_0^1 f(x, y) \, dy \quad (0 \leq x \leq 1).$$

Is $g$ continuous?

10. Let $A$ be Lebesgue measurable subset of the reals; and suppose $\mu(A)$ is finite, where $\mu$ is Lebesgue measure. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \mu ((-\infty, x) \cap A).$$

(a) Is $f$ continuous on $\mathbb{R}$?
(b) Is $f$ uniformly continuous on $\mathbb{R}$?