First Year Exam - Analysis
Sept. 1991

All work must be presented in a neat and logical manner. Put each problem on a separate sheet. Print Name on each sheet.

1. (a) Suppose \( f : (0, 1) \rightarrow \mathbb{R} \) is uniformly continuous.
   Show that \( \{ f(\frac{1}{n}) \}_{n=1}^{\infty} \) converges.

   (b) Define \( g : (0, 1) \rightarrow \mathbb{R} \) by \( g(x) = \sin(\frac{\pi}{2x}) \). Show \( g \) is not uniformly continuous.

2. (a) Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by
   \[
   f(x) = \begin{cases} 
   0, & \text{if } x \text{ is irrational} \\
   \frac{1}{n}, & \text{if } x = \frac{m}{n}, \text{ where } n > 0 \text{ and g.c.d. } (m, n) = 1.
   \end{cases}
   \]
   Show \( f \) is continuous at each irrational number.

   (b) Define \( g : \mathbb{R} \rightarrow \mathbb{R} \) by
   \[
   g(x) = \begin{cases} 
   1, & \text{if } x \text{ is irrational} \\
   0, & \text{if } x \text{ is rational}.
   \end{cases}
   \]
   Is \( f + g \) Riemann integrable on \([0, 1]\)?

3. In a metric space \( X \), let \( E' \) denote the set of all limit points of a set \( E \subset X \).
   Show that \( E' \) is closed and that \( E' = (E)' \).

4. Investigate the convergence or divergence of \( \Sigma a_n \) if
   (a) \( a_n = \sqrt{n+1} - \sqrt{n} \)
   (b) \( a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} \)
   (c) \( a_n = (\sqrt{n} - 1)^n \).

5. Suppose \( a_n > 0 \) and \( \Sigma a_n \) diverges. Prove that \( \Sigma \frac{a_n}{1 + a_n} \) diverges.

6. Prove that \( e^x < \frac{1}{1-x} \) whenever \( x < 1 \) and \( x \neq 0 \).
7. Prove that
\[ \ln 2 = \sum_{n=1}^{\infty} \frac{1}{n+1} \left( \frac{1}{2} \right)^{n+1}. \]
(Hint: Use the identity \( \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \).)

8. Let \( f \) be a nonnegative measurable function defined on the measurable set \( E. \)
   Suppose \( \int_E f \, dm = 0. \)
   Prove that \( f(x) = 0 \) almost everywhere on \( E. \)

9. Let \( A \) be a Lebesgue measurable set. Let \( \epsilon > 0 \) be given. Prove that there exists an
   open set \( G \) such that \( G \supset A \) and \( m(G - A) < \epsilon. \)

10. Give an example of a sequence of Lebesgue integrable functions \( f_n, n = 0, 1, 2, \ldots \) such
    that \( \{f_n\}_{n=1}^{\infty} \) converges uniformly to \( f_0 \) on \( \mathbb{R}, \) but \( \int_{\mathbb{R}} f_n \neq \int_{\mathbb{R}} f_0. \)