1. Give an example of:
   a) A closed set that is not perfect.
   b) A perfect set that is not closed.
   c) A countable set with a countable set of limit points.

2. Show that a metric space is disconnected if and only if it contains a set that is both open and closed.

3. For each natural number \( n \), define the function
   \[
   f_n(x) = \frac{nx}{1 + n^4x^4}
   \]
   on the interval \([0, 1] \subset \mathbb{R}\).
   a) Does the sequence \( \{f_n(x)\}_{n \in \mathbb{N}} \) converge uniformly on \([0, 1]?\)
   b) Does the sequence \( \{f_n(x)\}_{n \in \mathbb{N}} \) converge uniformly on any subintervals of \([0, 1]?\) If so, what are they?

4. Given a continuous function \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \int_{-a}^{a} f(t)t^{2m}dt = 0 \) for all \( m \in \mathbb{N} \), show that \( f \) is an odd function on the interval \([-a, a]\).

5. Let \( D \) be a compact, connected region in \( \mathbb{R}^2 \). Let \( f \) and \( g \) be continuous, bounded, real-valued functions on \( D \) with \( g(x) \geq 0 \) for all \( x \in D \). Show that there is a point \( x_0 \in D \) such that
   \[
   \iint_D fg \, d^2x = f(x_0) \iint_D g \, d^2x.
   \]
   Can the assumption \( g(x) \geq 0 \) be removed without changing the conclusion? If yes, prove it. If not, then give a counterexample.

6. Assume that \( \sum_{n=1}^{\infty} \frac{a_n^2}{n} \) converges, where \( \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{R} \). Does the limit \( \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n \) exist? If so, what is it?
7. Let 
\[ f(x) = \begin{cases} 
  x^2, & \text{if } x \in \mathbb{Q} \text{ (the rationals)} \\
  0, & \text{if } x \not\in \mathbb{Q}. 
\end{cases} \]

a) Does \( f'(0) \) exist?
b) Does \( f'(x) \) exist for \( x \neq 0 \)?
c) Find those \( x \) where \( f \) has discontinuities of the first kind, of the second kind.

8. Let \([x]\) denote the largest integer less than or equal to \( x \) (the integer part of \( x \)), and let \( \alpha(x) = x[x^2] \). Find \( \int_0^2 x^2 \, d\alpha(x) \).

9. Let \( C^1(\mathbb{R}) \) be the set of all real-valued functions on \( \mathbb{R} \) that are continuously differentiable and have compact support.
   a) Show that \( C^1(\mathbb{R}) \) is not an empty set.
   b) Is \( C^1(\mathbb{R}) \) dense in \( L^2(\mathbb{R}, dx) \)? Prove it.

10. With \( f \) and \( g \) complex-valued functions in \( L^2(X, \mu) \), are the following two conditions equivalent?
   (i) \( |\int fg^* \, d\mu|^2 = (\int |f|^2 \, d\mu)(\int |g|^2 \, d\mu) \)
   (ii) There exists a constant \( \lambda \) such that \( f = \lambda g \) almost everywhere.