1st Year Analysis Examination
May 21, 1990

Do each problem on a separate sheet and write your name on each sheet. Also provide a cover page listing the problems you have attached. Each problem is worth 10 points, and you will be graded on a total of 100 points.

1. Suppose $f: [0, \infty) \to \mathbb{R}$ is continuous. Is it true that if $\lim_{x \to \infty} f(x) = 0$, then $f$ is uniformly continuous.

2. Find all values of $z$, in the complex plane, for which the series

$$\sum_{n=1}^{\infty} \frac{(z + 2)^{n-1}}{(n+1)^3 4^n}$$

converges. Draw a picture of your result.

3. Let $X$ (resp. $Y$) be the space of all continuous (resp. continuously differentiable) real-valued functions on the closed interval $[0,1]$; and for every pair $(f,g)$ of functions in $X$, let

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

(a) Is $(X,d)$ a complete metric space? (explain)

(b) Is $(Y,d)$ a complete metric space? (explain)

4. Suppose $f:(0,1) \to \mathbb{R}$ is differentiable. Is it true that if $f'$ is monotonic, then $f'$ is continuous.
5. Suppose that $f(x) \geq 0$ and that $f$ decreases monotonically for $x \geq 1$. Show that the integral

$$\int_{1}^{\infty} f(x) \, dx$$

converges if and only if

$$\sum_{n=1}^{\infty} f(n)$$

converges.

6. Let $\{f_n\}$ be an equicontinuous sequence of functions on $[0,1]$. Suppose that $\{f_n\}$ is pointwise convergent on $[0,1]$. Is $\{f_n\}$ uniformly convergent on $[0,1]$? Prove this fact, or give a counterexample.

7. Let $E$ be a Lebesgue measurable subset of $[0,1]$. Let $m$ denote Lebesgue measure, and suppose $0 \leq \alpha \leq m(E)$. Prove that there exists a Lebesgue measurable set $F \subseteq E$ such that $m(F) = \alpha$. [Hint: Consider the function $f(x) = m(E \cap [0,x]), 0 \leq x \leq 1$]

8. True or false: If $f$ is a nonnegative function defined on $\mathbb{R}$ and

$$\int_{\mathbb{R}} f \, dx < \infty$$

then $\lim_{|x| \to \infty} f(x) = 0$?

Justify your answer.
9. Let \( f : [0,1] \rightarrow \mathbb{R} \) be a Lebesgue measurable function. Prove that there exists a Borel measurable function \( g \) such that \( g = f \) a.e. on \([0,1]\) with respect to Lebesgue measure.

[Hint: Prove first when \( f \) is a simple function. For the general case, express \( f \) as the limit of simple functions].

10. Let \( f \) be a nonnegative measurable function defined on \( \mathbb{R} \). Prove that if

\[
g(x) = \sum_{n=-\infty}^{+\infty} f(x + n)
\]

is integrable, then \( f = 0 \) a.e.