First-Year Analysis Examination  
August 2016 Part Two

Answer exactly FOUR questions. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let the function $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. For each positive integer $n$ let $f_n(t) = f(t + \frac{1}{n})$ whenever $t$ is a real number. Prove that the sequence $(f_n)_{n=1}^\infty$ is uniformly convergent. Give an example (with justification) to show that the conclusion can fail if the hypothesis uniformly is dropped.

2. Let $A = \{a_n : n \geq 1\}$ be a countably infinite subset of $[0, 1]$ and let $1_A : [0, 1] \to \mathbb{R}$ be its indicator (or characteristic) function. Exhibit such a set $A$ for which $1_A$ is not Riemann integrable and exhibit such a set $A$ for which $1_A$ is Riemann integrable, providing justification in each case.

3. Let $(f_n)_{n=1}^\infty$ be a sequence of continuous functions from $[0, 1]$ to $[0, 1]$. Prove that if $f_n(t)$ decreases to 0 whenever $0 \leq t \leq 1$ then $\int_0^1 f_n(t) \, dt \to 0$. Does the same conclusion follow when decreases is replaced by converges? Justify.

4. Prove that if the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable then its derivative is measurable.

5. Let $f$ be a Lebesgue integrable function on $\mathbb{R}$ and let $g$ be defined on $\mathbb{R}$ by $g(y) = \int_{-\infty}^{\infty} \cos(xy)f(x) \, dx$. Prove that $\lim_{k \to \infty} g(k) = 0$. 