First-Year Analysis Examination, Part Two
January 2016

Do exactly two problems from Part A and two problems from part B. Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Part A

1. Let \( f_n : X \to \mathbb{R} \) be a uniformly convergent sequence of continuous functions on a compact metric space \( X \). Prove that the set \( \{f_n\} \) is equicontinuous on \( X \).

2. Let \( \sum_{n=0}^{\infty} a_n x^n \) be a power series which converges for all \( x \in \mathbb{R} \). State and prove the theorem concerning the uniform convergence of the series on finite intervals \([a, b] \). Must the series converge uniformly on all of \( \mathbb{R} \)? Prove, or give a counterexample.

3. Suppose \( f \geq 0 \), \( f \) is continuous on \([a, b] \), and \( \int_a^b f(x) \, dx = 0 \). Prove that \( f(x) = 0 \) for all \( x \in [a, b] \).

Part B

1. Let \( f_n : [0, 1] \to \mathbb{R} \) be a sequence of continuous functions. Prove that \( g = \lim \sup f_n \) and \( h = \lim \inf f_n \) are Lebesgue measurable.

2. For each of the following, either prove or give a counterexample (\( m \) denotes Lebesgue measure on \( \mathbb{R} \); assume all functions are measurable):
   a) If \( f_n \) is integrable for all \( n \), \( f_n \to f \) uniformly on \( \mathbb{R} \) and \( f \) is integrable, then \( \int_{\mathbb{R}} f_n \, dm \to \int_{\mathbb{R}} f \, dm \).
   b) If \( f_n \to f \) uniformly on \([0, 1] \) and \( f \) is integrable, then \( \int_0^1 f_n \, dm \to \int_0^1 f \, dm \).
   c) Suppose \( f_n \geq 0 \) and \( \int_0^1 f_n \, dm = 1 \) for all \( n \). If \( f_n \to f \) pointwise, then \( \int_0^1 f \, dm \leq 1 \).

3. State the monotone convergence theorem and Fatou’s theorem, and use the former to prove the latter.