Do exactly 2 problems from Part A and 2 problems from Part B. Work must be presented in a neat and logical fashion in order to receive credit. Do not leave any gaps. When a theorem is used in a proof it must be precisely stated.

**Part A**

1. For \( n \geq 1 \) and \( 0 < t < 1 \), let \( f_n(t) = t^n \). Prove that:
   a) \((f_n)\) converges uniformly on each compact subset of \((0,1)\),
   b) \((f_n)\) does not converge uniformly on \((0,1)\).

2. Suppose \( \mathcal{F} \) is an equicontinuous family of real-valued functions on \([0,1]\), such that
   \[
   \sup_{f \in \mathcal{F}} |f(0)| = M < +\infty.
   \]
   Prove that \( \mathcal{F} \) is uniformly bounded (that is, that \( \sup_{f \in \mathcal{F}} \sup_{x \in [0,1]} |f(x)| < +\infty \)).

3. Prove that every monotone function \( f : [a,b] \to \mathbb{R} \) is Riemann integrable.

**Part B**

1. State Fatou’s lemma and use it to prove the Dominated Convergence Theorem.

2. Let \( (f_n) \) be a sequence of real-valued measurable functions on a measurable space \((X, \mathcal{M})\). Prove that each of the following subsets of \( X \) is measurable:
   a) \( \{ x \in X : \text{the sequence } (f_n(x)) \text{ is unbounded} \} \).
   b) \( \{ x \in X : \text{the sequence } (f_n(x)) \text{ is strictly increasing} \} \).

3. Let \( g : [0,1] \times [0,1] \to \mathbb{R} \) be a function with the following properties:
   i) \( |g(x,t)| \leq 1 \) for all \( x \) and \( t \),
   ii) for each \( t \), the function \( x \to g(x,t) \) is continuous on \([0,1] \), and
   iii) for each \( x \), the function \( t \to g(x,t) \) is continuous on \([0,1] \).

   Prove that the function \( h \) defined by
   \[
   h(x) = \int_0^1 g(x,t) \, dt
   \]
   is continuous on \([0,1] \).