MAA 5228 First-Year Exam. May 2016

Do exactly four problems. Work should be presented in a logical fashion in order to receive full credit.

1. Let \( f \) be a continuous function on \([0, 1]\). Prove that \( f \) is uniformly continuous.

2. Let \( \{G_n\}_{n=1}^{\infty} \) be a family of open sets in \( \mathbb{R}^n \) and \( \{F_n\}_{n=1}^{\infty} \) a family of closed sets. Prove that \( \bigcup_{n=1}^{\infty} G_n \) is open and \( \bigcup_{n=1}^{N} F_n \) is closed for any finite \( N \). Give an example such that \( \bigcup_{n=1}^{\infty} F_n \) is not closed.

3. Suppose \( \sum_{n=1}^{\infty} a_n \) is an absolutely convergent real series. Let \( b_n \) be a re-arrangement of \( a_n \). Prove that \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n \).

4. Let \( f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } x=y=0. \end{cases} \) Prove that \( f \) is not continuous at \((0,0)\).

5. Suppose \( f'(x) > 0 \) for \( x \in (0, 1) \) and let \( g \) be the inverse function of \( f \). Prove that for \( x_0 \in (0, 1) \), \( g'(f(x_0)) = 1/f'(x_0) \).