First Year Algebra Exam

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Give an example of each of the following, with some explanation.
   (a) A group $G$ and a subgroup $H \leq G$ such that $C_G(H) \neq N_G(H)$.
   (b) A group $G$ and a subgroup $H \leq G$ such that $H \not\subseteq C_G(H)$.
   (c) A group $G$ and a normal subgroup $H \trianglelefteq G$ such that $C_G(H) \neq G$.

2. Let $G$ be a group and let $T_G = \{x \in G : x^n = 1 \text{ for some } n \geq 1\}$.
   (a) Prove that if $G$ is abelian then $T_G$ is a subgroup of $G$.
   (b) Give an example of a nonabelian group $G$ such that $T_G$ is not a subgroup of $G$. Justify any claims that you make.

3. Let $G$ be a finite group which acts on the set $S$. Let $H$ be a normal subgroup of $G$ with the following property: For every $x_1, x_2 \in S$, there is a unique $h \in H$ such that $h(x_1) = x_2$. Choose $x \in S$ and let $G_x = \{g \in G : g(x) = x\}$ denote the stabilizer of $x$.
   (a) Prove that $G = G_xH$ and $H \cap G_x = 1$.
   (b) Show that if $H \leq Z(G)$ then $G_x \trianglelefteq G$ and $G$ is the internal direct product of $G_x$ and $H$.

4. (a) Prove that every simple subgroup of $S_4$ is abelian.
   (b) Use the result from (a) to prove that if $G$ is a nonabelian simple group then every proper subgroup of $G$ has index at least 5.

5. Let $G$ be a group of order $76 = 4 \cdot 19$.
   (a) Prove that $G$ contains a normal Sylow 19-subgroup.
   (b) Prove that the center of $G$ contains an element of order 2.
6. Let $G$ be a group.

(a) Define the derived series $G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \ldots$ of $G$.
(b) Prove that $G$ is solvable if and only if $G^{(n)} = \{1\}$ for some $n \geq 0$.
(c) Give an example of a solvable group $G$ such that $G^{(1)} \neq \{1\}$.

7. Let $R$ be a commutative ring with identity, and denote by $N$ the set of nilpotent elements of $R$, i.e. the set of $x \in R$ such that $x^n = 0$ for some $n$ (possibly depending on $x$).

(a) Show that $N$ is an ideal of $R$.
(b) Show that 0 is the only nilpotent element of $R/N$.

8. Give an example of:

(a) a skew field that is not a field;
(b) a commutative ring with identity that is not an integral domain;
(c) an integral domain that is not a unique factorization domain;
(d) a unique factorization domain that is not a Euclidean domain.

9. Let $R$ be an integral domain.

(a) Define what it means for an element $x \in R$ to be irreducible, and what it means for $x \in R$ to be prime.
(b) Show that a prime element of $R$ is irreducible.
(c) Give an example of an integral domain $R$ and an $x \in R$ which is irreducible, but not prime.

10. Let $K = \mathbb{Q}(i, \sqrt{-2})$

(a) Find the degree of $K$ over $\mathbb{Q}$, and over $\mathbb{Q}[i]$.
(b) Show that if $\alpha = i + \sqrt{-2}$, then $K = \mathbb{Q}(\alpha)$.
(c) Show that $K$ does not contain an extension of $\mathbb{Q}$ of degree 3.

11. (a) Find representatives for each of the conjugacy classes in $GL_4(\mathbb{C})$ of order 3. (b) Do the same for $GL_4(\mathbb{F}_3)$. 