1. (i) (3 points) State the first isomorphism theorem for groups. (ii) (7 points) Use the first isomorphism theorem to prove the second isomorphism theorem.
2. (i) (3 points) Find representatives for all conjugacy classes of order 4 in $S_6$. (ii) (3 points) Do the same for $A_6$. (iii) (4 points) What is the cardinality of each of these classes?
3. (10 points) Classify all groups of order 21, constructing examples for each isomorphism class.
4. (10 points) Suppose $p$ is a prime, $G$ is a $p$-group, and $S$ is a set on which $G$ acts. Denote by $S^G$ the set of elements of $S$ fixed by the action, i.e. $x \in S$ if and only if $gx = x$ for all $g \in G$. Show that $|S^G|$ is congruent to $|S|$ modulo $p$. 

5. Recall that a subgroup $H$ of a group $G$ is characteristic if $\varphi(H) = H$ for all automorphisms $\varphi$ of $G$. (i) (3 points) Show that if $H$ is a characteristic subgroup of $G$, it is normal. (ii) (3 points) Show that if $K$ is a characteristic subgroup of $H$ and $H$ is a normal subgroup of $G$, then $K$ is a normal subgroup of $G$. (iii)(4 points) Give an example of a group $G$ with subgroups $K \subseteq H \subseteq G$ with $K$ normal in $H$, $H$ normal in $G$ but $K$ not normal in $G$. 
6. Let $R$ be a ring with identity (not necessarily commutative) and $M$ a left $R$-module. (i) (5 points) Show that the annihilator $\text{Ann}(M)$ of $M$ is a 2-sided ideal of $R$. (ii) (5 points) If $I$ is a right ideal of $R$, show that the set of $m \in M$ such that $Im = 0$ is a submodule of $M$. 
7. (10 points) Suppose $F$ is a field. Show, using Zorn’s lemma, that if $V$ is an $F$-vector space and $S$ is a set of linearly independent elements in $V$, then $S$ extends to a basis of $V$. 
8. (i) Show that if the field extensions $K/F, L/K$ are finite then so is $L/F$. (ii) Do the same with “finite” replace by “algebraic.”
9. (i) (5 points) What is the degree over \( \mathbb{Q} \) of \( \sqrt{2} + \sqrt{3} \)? Find the minimal polynomial of this element over \( \mathbb{Q} \). (ii) (2 points) What is the degree over \( \mathbb{Q} \) of a root in \( \mathbb{C} \) of the equation \( X^7 + 9X + 6 \)? (iii) (3 points) Find all irreducible polynomials in \( \mathbb{F}_2[X] \) of degree 4.
10. Give an example of (i) (2 points) a skew field that is not a field, (ii) (2 points) a commutative ring with identity that is not an integral domain, (iii) (3 points) an integral domain that is not a UFD, (iv) (3 points) a UFD that is not a PID. Explain your examples.
11. (i) (5 points) Find representatives for all conjugacy classes of order seven in $GL_4(\mathbb{F}_2)$. (ii) (5 points) Do the same for conjugacy classes of order 4, expressing the results in Jordan normal form.