Name: ______________________

1. (10 points) Determine the number of elements of order 6 in the symmetric group on 10 symbols.

2. (a) (5 points) Give an example of an abelian group with no maximal subgroups.
   (b) (5 points) Give an example of a commutative ring with no maximal ideals.

3. Let $G$ be a finite group, $H$ a normal subgroup. Suppose that $p$ is a prime and $Q$ is a Sylow $p$-subgroup of $H$.
   (a) (5 points) Prove the Frattini argument: $G = HN_G(Q)$.
   (b) (5 points) Suppose $K$ is a normal subgroup of $G$ of order prime to $p$, and that $P$ is a Sylow $p$-subgroup of $G$. Show that $N_{G/K}(PK/K) = N_G(P)K/K$. (Hint: Apply (a) to the group $N = N_G(PK)$)

4. Let $G$ be a group of order $p^n$, where $p$ is prime and $n \geq 1$.
   (a) (5 points) Prove that $G$ has a normal subgroup of order $p^m$ for every integer $m$ with $0 \leq m \leq n$. (You may use the fact that the center of a nontrivial $p$-group is not trivial.)
   (b) (5 points) Is it true that $G$ has a characteristic subgroup of order $p^m$ for every integer $m$ with $0 \leq m \leq n$? (Justify your answer.)

5. (10 points) Let $V$ be a finite-dimensional vector space over a field $F$ and $T \in \text{End}_F(V)$ a linear transformation. Prove that there is a unique monic polynomial $m(x)$ in $F[x]$ with the property that $m(x)$ has the minimum degree among nonzero polynomials $f(x)$ such that $f(T)$ is the zero transformation.

6. (10 points) A commutative ring $R$ with 1 is called a local ring if and only if it has a unique maximal ideal. Suppose that $R$ is ring with a proper ideal $M$. Show that $R$ is local, with maximal ideal $M$, if and only if every element of $R \setminus M$ (set difference) is a unit.

7. (a) (4 points) Prove that a prime $p \in \mathbb{N}$ such that $p \equiv 3 \pmod{4}$ is prime when considered as an element in the ring of Gaussian integers $\mathbb{Z}[i]$.
   (b) (3 points) Prove that a prime $p \in \mathbb{N}$ such that $p \equiv 1 \pmod{4}$ is a product of two non-associate primes in $\mathbb{Z}[i]$.
   (c) (3 points) Give a factorization of 30 as the product of a unit and powers of non-associate primes in $\mathbb{Z}[i]$.

8. Let $R$ be the ring of $n \times n$ matrices over a field $F$, where $n \geq 2$. 
(a) (7 points) Show that $R$ has no ideals other than 0 and $R$.
(b) (3 points) Exhibit a nonzero proper left ideal of $R$.

9. (10 points) Let $F$ be a field and $K$ a splitting field over $F$ of an irreducible polynomial $f(x) \in F[x]$. Prove that if $\alpha$ and $\beta$ are roots of $f(x)$ in $K$, then the two subfields $F(\alpha)$ and $F(\beta)$ of $K$ are isomorphic.

10. (10 points) Let $F$ be a field.
   (a) (3 points) What does it mean to say that a field extension of $F$ is algebraic?
   (b) (7 points) Let $F \subset K \subset L$ be three fields. Prove that if $K$ is algebraic over $F$ and $L$ is algebraic over $K$, then $L$ is algebraic over $F$.

11. (10 points) Find one representative of each conjugacy class of elements of order 4 in the group $GL(6, \mathbb{Q})$. 