First Year Algebra Exam – May 2007

Time allowed: 240 minutes

Do seven of the following ten problems. Please do not turn in more than seven problems.

You must show your work. Answers with no work and/or no explanations will receive no credit. State clearly any theorem you use in your proofs.

In the problems, $\mathbb{Z}$, resp. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{C}$, is the set of all integers, resp. positive integers, rational numbers, complex numbers.

1. State the three Sylow theorems and prove the existence of Sylow subgroups.

2. Let $H$ be a normal subgroup of $G$. Assume $H$ is cyclic. Prove that any subgroup of $H$ is also normal in $G$.

3. Classify, up to isomorphism, the groups of order 28 with non-cyclic Sylow 2-subgroups.

4. Prove that any nonzero vector space over a field $\mathbb{F}$ has a basis. (You may assume Zorn's Lemma).

5. Let $R = \mathbb{Z}[x]$ be the ring of polynomials in variable $x$ with integer coefficients. Let $I = (5, x^2 + 2)$ be the ideal of $R$ generated by 5 and $x^2 + 2$. Prove that $R/I$ is a finite field, and find its cardinality.

6. Consider the ring $\mathbb{F}[x, y]$ of polynomials in two commuting variables $x, y$ over a field $\mathbb{F}$. Is this ring (i) a Euclidean domain? (ii) a principal ideal domain? (iii) a unique factorization domain? Justify your answer(s).

7. Let $A$ be an $n \times n$-matrix with entries in $\mathbb{Q}$. Suppose $A$ is invertible and $A^4 - 4A = 2A^{-1}$. Show that $n$ is divisible by 5 and that such a matrix $A$ is unique up to similarity.

8. Let $T$ and $S$ be linear transformations $\mathbb{C}^5 \rightarrow \mathbb{C}^5$, both with the same characteristic polynomial $x^2(x^2 + 1)(x - 3)$. Assume that their kernels, $\text{Ker}(T)$ and $\text{Ker}(S)$, have the same dimension. Are $T$ and $S$ necessarily similar? Justify your answer.

9. Let $p \in \mathbb{N}$ be a prime. Find a splitting field $K$ for $x^p - p$ over $\mathbb{Q}$, and determine $[K : \mathbb{Q}]$.

10. Let $\mathbb{F}$ be a subfield of a field $K$ and let $\alpha, \beta \in K$ be algebraic over $\mathbb{F}$, of degree $m$ and $n$, respectively. Assume $m$ and $n$ are coprime. Find the degree $[\mathbb{F}(\alpha, \beta) : \mathbb{F}]$. 

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