First Year Algebra Exam – Sept. 2002

Time allowed: 240 minutes

Do seven of the following ten problems. Please do not turn in more than seven problems. The problems do not have to be attempted in the order they are listed.

You must show your work. Answers with no work and/or no explanations will receive no credit. State clearly any theorem you use in your proofs.

In the problems, $\mathbb{Z}$, resp. $\mathbb{Q}$, $\mathbb{C}$, is the set of all integers, resp. rational numbers, complex numbers.

1. State and prove Cayley's Theorem about finite groups.

2. Let $G$ be a group of order 44. Prove that
   (i) a Sylow 11-subgroup of $G$ is normal in $G$, and
   (ii) the center of $G$ contains an element of order 2.

3. Classify the groups of order $57 = 19 \cdot 3$ up to isomorphism.

4. Prove the Chinese Remainder Theorem: Let $R$ be a commutative ring with identity. Let $I, J$ be ideals of $R$ such that $I + J = R$. Then the homomorphism $\varphi : R \rightarrow R/I \times R/J$, $\varphi(a) = (a + I, a + J)$, is surjective, and its kernel is $I \cap J$.

5. Let $F[x, y]$ be the ring of (commutative) polynomials in two variables $x, y$ over a field $F$. Is $F[x, y]$ (i) A Euclidean domain? (ii) A principal ideal domain? (iii) A unique factorization domain? Justify your answer.

6. Using Eisenstein's criterion, prove that there are infinitely many irreducible monic polynomials of degree 2002 over $\mathbb{Q}$. Is the statement still true if one replaces $\mathbb{Q}$ by $\mathbb{C}$? Justify your answer.

7. Let $F$ be any field and $A, B$ any two $4 \times 4$-matrices over $F$. Using the rational canonical form, prove that the following two conditions are equivalent:
   (i) $A$ and $B$ are similar;
   (ii) $A$ and $B$ have the same characteristic polynomial, the same minimal polynomial, and the same number of invariant factors.

8. Let $V$ be a finite dimensional vector space over a field $F$. A linear transformation $T : V \rightarrow V$ is called a projection if $T^2 = T$. Prove that two projections $T, S : V \rightarrow V$ are similar if and only if their kernels have the same dimension.

9. Find a splitting field $K$ for $x^4 - 5$ over $\mathbb{Q}$, and determine $[K : \mathbb{Q}]$.

10. Let $F$ be an extension of degree 2002 of $\mathbb{Q}$. Prove that $F$ contains no cubic roots of 2.