First Year Algebra Exam – January 2002

Time allowed: 240 minutes

Do seven of the following ten problems. Please do not turn in more than seven problems. You must show your work. Answers with no work and/or no explanations will receive no credit. State clearly any theorem you use in your proofs.

In the problems, \( Z \), resp. \( Q \), \( C \), is the set of all integers, resp. rational numbers, complex numbers.

1. State and prove Cayley’s Theorem about finite groups.

2. Let \( H \) be a normal subgroup of \( G \). Assume \( H \) is cyclic. Prove that any subgroup of \( H \) is also normal in \( G \).

3. Prove that there is no simple group of order \( 2002 = 2 \times 7 \times 11 \times 13 \).

4. Consider the ring \( Z[x] \) of polynomials in one variable \( x \) over the integers. Is \( Z[x] \) (i) a Euclidean domain? (ii) a principal ideal domain? (iii) a unique factorization domain? Justify your answers.

5. Let \( R \) be a ring. An \( R \)-module \( M \) is called irreducible if \( M \neq 0 \) and 0 and \( M \) are the only submodules of \( M \).
   a) Prove Schur’s Lemma: Suppose the \( R \)-modules \( M \) and \( N \) are irreducible. Then every nonzero homomorphism \( T : M \to N \) is an isomorphism.
   b) Let \( M \) be an irreducible \( R \)-module. Deduce from a) that \( \text{End}_R(M) \), the set of all (module) homomorphisms \( M \to M \), is a division ring.

6. Let \( F \) be a field and \( G \) be a finite subgroup of \( F \setminus \{0\} \), the multiplicative group of \( F \). Prove that \( G \) is cyclic.

7. Let \( A \) and \( B \) be two \( 4 \times 4 \) matrices over \( C \), both with the same characteristic polynomial \( (x - 1)^2(x - 2)(x - 3) \). Assume that \( A \) and \( B \) have the same minimal polynomial. Are \( A \) and \( B \) necessarily similar? Justify your answer.

8. How many conjugacy classes of elements of order 4 are there in the group \( GL_3(Q) \) of invertible \( 3 \times 3 \) matrices over \( Q \)? Justify your answer and give a representative for each conjugacy class.

9. Find a splitting field \( F \) for \( x^6 - 4 \) over \( Q \), and find \( [F : Q] \).

10. Let \( F \) be a field and \( E \) be an extension of \( F \) of degree 2001. Prove that \( F(u) = F(u^2) \) for any \( u \in E \).