FIRST-YEAR EXAM - SECOND-SEMESTER TOPOLOGY - SPRING 2019

Answer all questions and work all problems. Make proofs as succinct as possible. Work each problem on a separate sheet of paper. Each problem is worth the points allotted.

Problem 1. (5 points) Let \( n \geq 1 \). Suppose that \( f, g : X \to \mathbb{R}^n \) are continuous. Show that \( f \) and \( g \) are homotopic.

Problem 2. (5 points) Let \( \gamma : [0,1] \to X \) with basepoint \( x_0 \). Let \( \gamma^{-1} \) be defined by \( \gamma^{-1}(t) = \gamma(1-t) \). Show that \( \gamma \ast \gamma^{-1} \) is loop homotopic to the constant loop \( 1 : [0,1] \to X \), \( 1(t) \equiv x_0 \).

Problem 3. (5 points) Show that \( \pi_1(S^1,1) = \mathbb{Z} \).

Problem 4. (5 points) Show that the disk, \( D^2 = \{ z \in \mathbb{C} \mid \|z\| \leq 1 \} \) has the fixed point property.

Problem 5. (5 points) Consider the matrix \( M = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \). This represents a group homomorphism \( M : \mathbb{Z}^2 \to \mathbb{Z}^2 \). Show that there is a map \( f : \mathbb{T}^2 \to \mathbb{T}^2 \) such that \( f_* = M \) where \( f_* : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2) \) is the homomorphism induced by \( f \) on the fundamental group.

Problem 6. (5 points) Suppose that \( f : S^1 \to S^1 \) is continuous. Show that \( f \) is homotopic to \( z^n : S^1 \to S^1 \) for some \( n \in \mathbb{Z} \). This is the degree of \( f \).

Problem 7. (5 points) Let \( n > 1 \). Let \( \mathbb{P}^n = S^n / D \) where \( D \) identifies \( x \) with its antipode for all \( x \in S^n \). Show that \( \pi_1(\mathbb{P}^n, x_0) \cong \mathbb{Z}_2 \).

Problem 8. (5 points) Describe space \( X \) such that \( \pi_1(X, x_0) \cong \mathbb{Z}_3 \). Verify that this is the fundamental group by the Seifert–van Kampen Theorem.

Problem 9. (5 points) A topological space is said to be Lindelöf provided that for every open cover \( \mathcal{U} \) of \( X \), there is a countable \( \mathcal{V} \subset \mathcal{U} \) covering \( X \). Show that if \( X \) is a separable metric space, then \( X \) is Lindelöf.
Problem 10. (5 points) State and prove the Tychonoff Theorem.

Problem 11. (50 points) State the following theorems or verify the following statements with quick examples or statements.

The Alexander Subbase Theorem.

The Hahn–Mazurkiewicz Theorem

The Brouwer Characterization of the Cantor Set

The Jordan Curve Theorem

The Borsuk-Ulam Theorem for $S^2$.

Determine the fundamental group of $T^2$.

Determine the fundamental group of the connected sum of two tori, $\pi_1(T^2 \# T^2)$.

The Arcwise Connectedness Theorem

Suppose that $X$ is a compact metric AR. Show that for any $Z$ and any pair of continuous functions $f, g : Z \to X$, $f$ and $g$ are homotopic.

Suppose that $C$ is the Cantor set. Show that $C$ is homeomorphic to $C \times C$. 