1. Assume that $A$ is Hermitian and all its eigenvalues are distinct and nonzero.
   (a) Show that each pair of distinct eigenvectors of $A$ are orthogonal.
   (b) If $\lambda$ is an eigenvalue of $A$ with eigenvector $x$ with $\|x\|_2 = 1$, then $B = A - \lambda xx^*$ has the same eigenvectors as $A$ while $B$’s eigenvalues are the same as those of $A$ except $\lambda$ is replaced with zero.

2. (a) If $P$ is a projector, prove that $\text{null}(P) \cap \text{range}(P) = \emptyset$ and $\text{null}(P) = \text{range}(I - P)$.
   (b) Prove that $P$ is an orthogonal projector if and only if it is Hermitian.

3. (a) If both $A$ and $U$ are in $\mathbb{C}^{m,m}$ and $U$ is unitary, prove that $\|UA\|_F = \|AU\|_F = \|A\|_F$.
   (b) If both $A$ and $U$ are in $\mathbb{C}^{m,m}$ and $U$ is unitary, prove that $\|UA\|_2 = \|AU\|_2 = \|A\|_2$.
   (c) Prove that $\|A\|_2 = (\rho(A^*A))^{1/2} = \sigma_1$, where $\sigma_1$ is the largest singular values of $A$.

4. (a) If $A \in \mathbb{C}^{m,n}$ with $m \geq n$, prove that $A^*A$ is invertible if and only if $\text{rank}(A) = n$.
   (b) Give an explicit formula for $\det(\lambda I - ww^*)$ when $\lambda \in \mathbb{C}$, $I$ is the $m \times m$ identity matrix and $w \in \mathbb{C}^m$.

5. Assume that $T$ is tridiagonal and symmetric with the diagonal entries given by $a_j$ for $j = 1, \ldots, m$ and the super- and sub-diagonal entries by $b_j$ for $j = 1 \ldots m - 1$. Let $p_k$ be the characteristic polynomial of the $k \times k$ matrix in the upper left hand corner of $A$. Prove that
   $$p_k(x) = (a_k - x)p_{k-1}(x) - b_{k-1}^2p_{k-2}(x).$$