MAA 5228 First-Year Exam, January 2019

Do exactly 4 problems. Work must be presented in a neat and logical fashion in order to receive credit. Do not leave any gaps. When a theorem is used in a proof it must be precisely stated.

1. Let \((s_n)_{n=1}^{\infty}\) be a bounded sequence of real numbers. Consider the sequence \((\sigma_n)\) of averages 
\[
\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.
\]
Prove that the sequence \((\sigma_n)\) is bounded and 
\[
\limsup \sigma_n \leq \limsup s_n.
\]

2. Let \(K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots\) be a decreasing family of nonempty sets in a metric space \(X\).
   a) Prove that if each \(K_n\) is compact, then \(\cap_n K_n\) is nonempty.
   b) Given an example to show that \(\cap_n K_n\) can be empty if the \(K_n\) are only assumed closed.

3. Let \(X, Y\) be metric spaces and \(f : X \to Y\) a function. Prove that if \(f\) is uniformly continuous on \(X\) and \((x_n)\) is a Cauchy sequence in \(X\), then \((f(x_n))\) is a Cauchy sequence in \(Y\).

4. Let \(X\) be a metric space and suppose that every continuous function \(f : X \to \mathbb{Z}\) is constant. Prove that \(X\) is connected.

5. Suppose \(f\) is differentiable in \((a, b)\) and \(f'(x) > 0\) for all \(x \in (a, b)\). Let \(g\) be the inverse function to \(f\). Prove that for each \(x_0 \in (a, b)\), the function \(g\) is differentiable at \(f(x_0)\) and 
\[
g'(f(x_0)) = \frac{1}{f'(x_0)}.
\]