Name: ______________________________

Problems to be graded: 1  2  3  4  5  6

1. (10 points) Let $p$ be a prime, let $G$ be a finite group, and let $P$ be a Sylow $p$-subgroup of $G$. Let $H$ be a subgroup of $G$ containing the normalizer $N_G(P)$ of $P$ in $G$. Prove that then $N_G(H) = H$.

2. (10 points) State and prove the Second Isomorphism Theorem for groups.

3. Let $n \geq 2$, and let $S_n$ be the symmetric group on $n$ letters.
   (a) (2 points) Define what is a transposition of $S_n$.
   (b) (2 points) Give an example of a transposition of $S_n$.
   (c) (6 points) Prove that every element of $S_n$ is a product of transpositions of $S_n$.

4. Let $G$ be a finite group, and assume that $G$ acts transitively on the finite non-empty set $\Omega$. Let $\omega \in \Omega$. We denote by $G_\omega$ the set of elements of $G$ that fix $\omega$.
   (a) (3 points) Prove that $G_\omega$ is a subgroup of $G$.
   (b) (7 points) Prove that the number of elements in $\Omega$ is exactly $[G : G_\omega]$.

5. Let $G$ be a finite group.
   (a) (5 points) Define what it means to say that $G$ is solvable.
   (b) (5 points) Without assuming any result about solvable groups (i.e. just from your definition), prove that if the order of $G$ is 20 then $G$ is solvable.

6. (10 points) Prove or disprove the following statement. Let $n$ be a natural number, let $S_n$ be the symmetric group on $n$ letters, let $A_n$ be the alternating group on $n$ letters, and let $\sigma \in S_n$. Then, if $\sigma$ has order 2019 then $\sigma \in A_n$. 